SCALING OF THE DYNAMICS OF VISCOELASTIC FLUIDS AND A CONSTRUCTAL APPROACH TO A FINITE VOLUME SPREADING

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Abstract:
The communication address transient dynamics of a finite volume of a viscoelastic fluid (block and droplets) driven by different forces applied on the body interfaces. The study is a natural continuation of the preceding results of the team [1, 2] with the application of Bejan's constructal concept and the discriminated dimensional analysis (DDA) of Huntley. Parallel to the results of DDA, scaling, and asymptotes from the differential model of viscoelastic (second-grade) fluid in transient motion (Stokes 'first problem), toward the evaluation of characteristic scales and dimensionless groups. The synergy of three different approaches allows us to go beyond the Newtonian problems studied in [1,2] and explore the power of the constructal theory, DDA, and model scaling in problems related to non-Newtonian fluids exhibiting relations.

Keywords:
Scaling, Viscoelasticity, Deborah number, Discriminate dimensional analysis, Constructal theory

1. Introduction

This correspondence and several recent publications of the team [1,2] have been inspired by the works of A. Bejan [3], who referenced the constructal law and demonstrated how flow configurations developed to enhance momentum transfer. Similarities to boundary layer configuration as asymptotic of time-evolution of a finite fluid volume were predicted in Prandtl's boundary layer theory [4, 5,6]. The distinction between the method and the theory of the is essential.
Beyond the results developed in [1,2], where Newtonian fluids were considered, we now address a case with a non-Newtonian second-grade viscoelastic fluid [7,8,9] as it is described next.

2. The problem

2.1. Finite fluid volume time evolution

Consider a fixed volume of viscoelastic fluid, schematically represented as a rectangle with height H and length L, at rest on a flat surface, and suddenly driven by a change in the velocity of its top surface $U = U_0$ (the same as in Stokes' first problem [10]). We are interested in the time evolution of such a fluid volume while preserving volume (i.e., in the two-dimensional case, this means preserving the area $A = L_0 H_0 = LH = const$, in time and the dimensional analysis of its transient flow).

This communication will consider the problem in its simplified form, which pertains to the spreading of viscoelastic droplets on flat surfaces.

This can be clarified by examining the temporal evolution of a finite volume, which gradually reduces over time to a thin layer of nearly infinite length; in this context, we will draw parallels with established concepts from classical hydrodynamics, specifically the boundary layer.

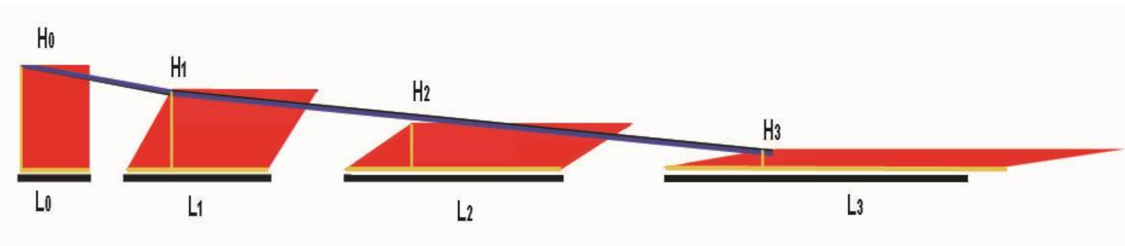

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Figure 1. A schematic representation of the fluid volume evolution in time toward a slender layer along a flat surface

2.2. Problem solution strategy

Before commencing dimensional analysis and a more comprehensive examination, it is essential to address several critical issues about the dynamics of viscoelastic flow and the resultant dimensionless number that governs it. This method entails several steps, including:

1. Constitutive equation of second-grade viscoelastic fluids and the scaling of the one-dimensional model equation, revealing the resulting dimensionless groups (Section 2).
2. Model scaling allows for the detection of the dimensionless groups controlling the governing equation (Section 3).
3. The approximate solution of the first Stokes' problem illustrates how the established dimensionless group influences the flow field and highlights the specific contributions of the elastic and viscous properties of the fluid (Section 4)
4. Discriminate dimensional analysis to the problem formulated by Figure 1, allowing us to attain results with the same physical meaning as those from the approximate solution, and last, but not least, to establish the boundary layer similarities as it was done in [1,2] for Newtonian fluid (Section 6).

3. The second-grade (viscoelastic) fluid models: the necessary background

3.1. Second-grade (viscoelastic) fluid: Constitutive equations

For the second-grade fluid, the Cauchy stress tensor $\mathbf{T}$ can be expressed as [7,8,9,10,]

$$\mathbf{T} = -\rho\mathbf{I} + \mu\mathbf{A} + \alpha\mathbf{A} + \alpha\mathbf{A} \tag{1}$$

where ρ is the density, $\mathbf{I}$ is the unit vector, α and α are the normal stress moduli, $\mathbf{A}$ and $\mathbf{A}$ are the kinematical tensors defined as [1]

$$\mathbf{A} = grad\,\mathbf{V} + \left(grad\,\mathbf{V}\right), \quad \mathbf{A} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{A}\left(grad\,\mathbf{V}\right) + \left(grad\,\mathbf{V}\right)\mathbf{A} \,. \tag{2}$$

These are constructive differential equations. For the second-grade fluids with a stress tensor expressed by Eq. (1), which is thermodynamically compatible, the following restrictions of the material moduli hold [1, 2].

$$\mu \geq 0\,, \alpha \geq 0\,, \; \alpha + \alpha = 0\,. \tag{3}$$

In the absence of body forces, the momentum and continuity equations are

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbf{T} \ , \quad \nabla \cdot \mathbf{V} = 0 \ . \tag{4}$$

Commonly, the constitutive equation for second-grade liquid is expressed in two forms [1,3]

$$\tau(t) = \mu\varepsilon(t) + E\frac{d\varepsilon(t)}{dt} \tag{5}$$

Moreover, $T = T = T = T = T = 0$, where $T = T$.

This model corresponds to the Maxwell spring-snapshot mechanical equivalent with an elastic element and a Newtonian viscous damper in a series, as it is shown in Figure 2.

Figure 2. Maxwell spring-dashpot mechanical model

3.2. One-dimensional model

Considering a transient motion of a second-grade semi-infinite fluid due to a sudden change in the surface velocity to U (parallel to the x axis), we refer to the so-called Stokes' first problem and follow the constitutive equation (5), as well as (2a), and we get the following constitutive equation:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial u}{\partial y} + \beta\frac{\partial u}{\partial t \partial y} \quad , \ \beta = \nu\lambda \, . \tag{6}$$

The boundary and initial conditions

$$u(y,0) = 0 \, , \ y > 0 \, ; \quad u(0,t) = U \ , \ t > 0 \, ; \quad u \to 0 \, , \ y \to \infty \tag{7}$$

In detail, in (6a) $\nu = \mu/\rho$ is the kinematic viscosity of the fluid $[m/s]$, where λ is the relaxation time $[s]$.

4. Model scaling

Here, we apply two approaches in the model scaling, leading to different results and enabling different interpretations of the transient behavior of the fluid. Precisely, these two approaches are pertinent to the velocity scale $U_{ref} = U_0$: 1) In the first approach, we use a certain velocity scale U_0 , but the scaling leads to a completely dimensionless equation; 2) In the second attempt, we will see the appearance of the Weisenberg (Wi) and Reynolds (Re) number, respectively. With both scaling approaches and the approximate solutions based on them, we may extract full information about the physics of transient fluid motion.

4.1. Scaling approach 1

Before developing any particular cases, we have to define the characteristic time and length scale, and we will do that based on the local (integer-order) model, since it is more familiar to the scientific society.

First, we consider that the variable u is dimensionless $u = U/U_0$, where U_0 is a certain reference value defined by the framework of the problem solved (in Stokes' first problem, the surface velocity U_0).

Thus, now we recognize that the dimension of ν is m/s as in the classical diffusion model of momentum, where there is no relaxation (i.e., with $\lambda = 0$). Taking into account that the time-dependent term on the left side of (6) has a dimension of $1/s$, then the question is about the dimensions of λ. A simple inspection reveals that the dimensional homogeneity of the equation requires β to have a dimension $[m]$, while the dimension of λ is $[s]$, so that $\beta = \nu\lambda$.

For the sake of clarity, let us write the first step of the nondimensionalization of the governing equation as

$$\frac{U \partial u}{t \partial \tilde{t}} = \nu \frac{U}{L} \frac{\partial u}{\partial \tilde{y}} + \beta \frac{1}{t} \frac{U}{L} \frac{\partial}{\partial \tilde{t}} \left(\frac{\partial u}{\partial \tilde{y}} \right) \tag{8}$$

with $u = U/U$, $\tilde{t} = t/t$, $\tilde{y} = y/L$.

This step leads to a completely dimensionless model (8)

$$\frac{\partial u}{t \partial \tilde{t}} = \nu \frac{1}{L} \frac{\partial u}{\partial \tilde{y}} + \beta \frac{1}{t} \frac{1}{L} \frac{\partial}{\partial \tilde{t}} \left(\frac{\partial u}{\partial \tilde{y}} \right) \tag{9}$$

Precisely, multiplying both sides of (8) by t we get

$$\frac{\partial u}{\partial \tilde{t}} = \frac{t}{\left(\frac{L}{\nu}\right)} \frac{\partial u}{\partial \tilde{x}} + \frac{t}{\left(\frac{L}{\nu}\right)} \frac{\lambda}{t} \frac{\partial}{\partial \tilde{t}} \left(\frac{\partial u}{\partial \tilde{x}} \right) \Rightarrow \frac{\partial u}{\partial \tilde{t}} = \frac{\partial u}{\partial \tilde{x}} + \frac{\lambda}{t} \frac{\partial}{\partial \tilde{t}} \left(\frac{\partial u}{\partial \tilde{x}} \right) \tag{10}$$

The dimensional homogeneity of (8) provides that the characteristic scales are $t = L/\nu$ and $L = \sqrt{\beta} = \sqrt{\nu\lambda}$. Then, the ratio λ/t is equal to unity because $\lambda/t = \frac{\lambda}{L/\nu} = \nu\lambda/L$. With $L = \sqrt{\beta} = \sqrt{\nu\lambda}$ we have $\lambda/t = \frac{\lambda}{L/\nu} = \nu\lambda/\nu\lambda = 1$. Consequently, if the length scale is $\sqrt{\beta} = \sqrt{\nu\lambda}$ as introduced above, thus *omitting the necessity to have a characteristic geometric length scale*, we can simplify the non-dimensional model (10) as

$$\frac{\partial u}{\partial \tilde{t}} = \frac{\partial u}{\partial \tilde{y}} + \frac{\partial}{\partial \tilde{t}} \left(\frac{\partial u}{\partial \tilde{y}} \right) \tag{11}$$

For example, λ can be considered as a relaxation time if we consider (6) as a result of the Taylor series expansion of the momentum flux, that is

$$\frac{\partial u}{\partial t} = \left(1 + \lambda \frac{\partial}{\partial t}\right)\left(\nu \frac{\partial u}{\partial x}\right) \Rightarrow \frac{\partial u}{\partial t} = \nu \frac{\partial u}{\partial x} + \nu\lambda \frac{\partial}{\partial t} \frac{\partial u}{\partial x}, \; \beta = \nu\lambda \; . \tag{12}$$

If we now need to define the dimensions of time $\tilde{t} = t/t$, where $t = \beta/\nu = \lambda$ this leads to $\tilde{t} = t/(\beta/\nu) \Rightarrow \tilde{t} = t/\lambda$, and we get a completely defined dimensionless time as a ratio of the process (observation) time t to the relaxation time λ ; this is the Deborah number [11,12] , as it will appear in the scaling and solution outcomes in the sequel.

This interpretation needs to define whether the process time scale is $t = L/\nu$ defined by the geometry of the process or if it is determined by a relaxation parameter λ. We have to mention that if the transient process is considered as taking place in a semi-infinite medium, then, since there is no characteristic length scale, we may use $\sqrt{\nu t}$ and therefore $\tilde{y} = y/\sqrt{\nu t}$, the Boltzmann similarity variable.

The example discussed later in this text will provide more specific interpretations of the scaling parameters and their definitions.

4.2. Scaling approach 2

Let us represent eq. (8), precisely its 3rd-order term, as

$$\frac{\partial u}{\partial \tilde{t}} = \frac{t}{\left(\frac{L}{\nu}\right)} \frac{\partial u}{\partial \tilde{y}} + \left(\frac{\lambda U}{L}\right)\left(\frac{\nu}{LU}\right) \frac{\partial}{\partial \tilde{t}}\left(\frac{\partial u}{\partial \tilde{y}}\right) \tag{13}$$

Now, we can identify three dimensionless ratios:

1) The ratio $\nu t / L$ that leads to the diffusional time scale $t = L/\nu$ in its classical definition.
2) The Weissenberg number $Wi = \frac{\lambda U}{L}$, also known as *the viscoelastic ratio number*, represents the comparison of the elastic to the inertial forces. (including viscous effect) forces [11,13].

To be clear, the Weissenberg number is defined as $Wi = 2\frac{\lambda \mu \dot{\gamma}}{\mu \dot{\gamma}} = 2\lambda \dot{\gamma}$; the characteristic deformation rate $\dot{\gamma} = \frac{\partial U}{\partial y}$, which *should be scaled through the characteristic length* L and U as a velocity scale.

3) The Reynolds number in its classical definition $\frac{\nu}{LU} = \frac{1}{\mathrm{Re}}$.

Hence, the scaled model (12) can be expressed as

$$\frac{\partial u}{\partial \tilde{t}} = \frac{\partial u}{\partial \tilde{y}} + \frac{Wi}{\mathrm{Re}} \frac{\partial}{\partial \tilde{t}}\left(\frac{\partial u}{\partial \tilde{y}}\right) \tag{14}$$

With one step ahead, recalling that the ratio $\frac{Wi}{\mathrm{Re}} = St$ defines the Strouhal number [11], we get

$$\frac{\partial u}{\partial \tilde{t}} = \frac{\partial u}{\partial \tilde{y}} + St \frac{\partial}{\partial \tilde{t}}\left(\frac{\partial u}{\partial \tilde{y}}\right) \tag{15}$$

4. An instructive approximate solution with a physical sound

Now, we apply an approximate solution method [14, 15] to both versions of the scaled model equations and will see whether the above-defined dimensionless groups appear in the solutions and how they control the fluid velocity profile.

4.1. The main approach to the solutions: the physical background formulation

When the surface velocity of the volume suddenly changes from zero to $U = U_0$, the transport of momentum in the fluid depth, along the y axis, develops gradually, with a front $\delta(t)$, simultaneously exhibiting the influence of both the elasticity and the viscosity. During this first stage of flow field development, the fluid can be considered as semi-infinite. The flow field reaches full development when $\delta = H$, with the assumption that at $y = H$, the velocity is zero due to the non-slip condition. After this initial flow field development, the discrete fluid volume of interest starts to change its shape down to a slender layer, as we explained at the beginning. This formulation enables the development of an analytical solution that clarifies the primary controlling dimensionless groups and pertinent physical effects and subsequently establishes similarities via dimensional analysis applied to the discrete fluid volume evolving while maintaining its volume (mass) with constant density.

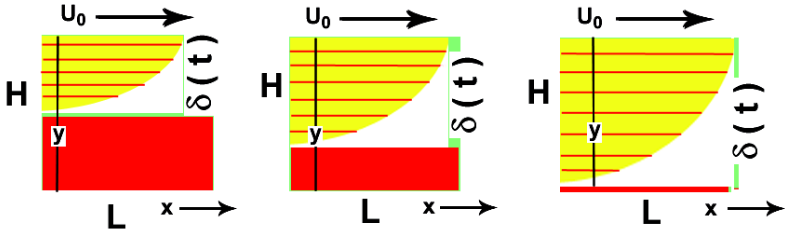

Figure 3. Flow field development schematically

4.2. A solution controlled by the Deborah number

Following the formulation done above, the fluid motion due to the sudden change in the surface velocity causes a flow field with a front of the disturbance propagation into the fluid depth, $\delta(t)$, where the following boundary conditions are obeyed (sharp front conditions) [14,16].

$$u_{y=\delta}=0 \text{ and } \left.\frac{\partial u}{\partial y}\right|_{y=\delta}=0. \tag{16}$$

Hence, the fluid volume is separated into two sections: disturbed for $0<y<\delta(t)$ and undisturbed (the red bottom layer in Figure 2) for $\delta(t)<y<H$.

For the sake of simplicity of the explanation, let us assume the velocity profile to be presented as a parabolic one satisfying the condition (15) [12].

$$u=u\left(1-\frac{y}{\delta}\right) \tag{17}$$

Applying double integration over the penetration depth [12]

$$\int\int\frac{\partial u(y,t)}{\partial t}dydy=-\int\nu\frac{\partial u(y,t)}{\partial y}dy+\beta\int\int\frac{\partial u(y,t)}{\partial t\partial y}dydy \tag{18}$$

And applying the Leibniz rule to the left-hand side of (17), we get

$$\frac{d}{dt}\left[\frac{\delta}{(n+1)(n+2)}-\beta\right]=\nu \tag{19}$$

And with the initial condition $\delta(t=0)=0$ we have from (18)

$$\delta=\sqrt{\nu t}\sqrt{(n+1)(n+2)}\sqrt{\left(1+\frac{\beta}{\nu t}\right)} \tag{20}$$

The term $\beta/\nu t$ is the Deborah number $De=\lambda/t$: the ratio of the relaxation time $\lambda=\beta/\nu$ to the observation time t. The expression (19) reveals that with an increase in time, the term $\beta/\nu t$ becomes negligible; that is, for large times, we have $\delta=\sqrt{\nu t}\sqrt{(n+1)(n+2)}$. Then, the velocity profile is

$$\frac{u}{U}=\left(1-\frac{y}{\sqrt{\nu t}\sqrt{(n+1)(n+2)}\sqrt{\left(1+\frac{\beta}{\nu t}\right)}}\right)=\left(1-\frac{\eta}{F\sqrt{(1+De)}}\right),\ \eta=\frac{y}{\sqrt{\nu t}}\quad . \tag{21}$$

i) As outcomes, we have three important results:

ii) The similarity variable $\eta = y/\sqrt{\nu t}$ is defined naturally in the case of transient Newtonian viscous flows.

iii) The approximate profile naturally defines the Deborah number in its classic form $De = \beta/\nu t$.

iv) The solution tends to the viscous Newtonian limits as the contribution of the 3rd-order term vanishes, which comes naturally with an increase in the observation time t , making the terms dependent on the Deborah number.

From the velocity profiles in Figure 3-(a) and (b) we see how the Deborah number, as a parameter, affects the velocity profiles in the depth of the fluid; $De = 1$ is a borderline where the relaxation time λ equals the observation time t, and both the elastic and viscous forces have equal orders of magnitude [17]; for $De > 1$ the elastic effects dominate, while for $De < 1$ the viscous forces dominate [17]. The three-dimensional plot in Figure 3-(c) clearly shows the simultaneous effects of these forces. Alternatively, we may define a new similarity variable

$$X=\frac{y}{\sqrt{\nu t}\sqrt{\left(1+\frac{\beta}{\nu t}\right)}}=\frac{y}{\sqrt{\nu t}\sqrt{\left(1+\frac{\lambda}{t}\right)}}=\frac{\eta}{\sqrt{(1+De)}} \tag{22}$$

For a long time, i.e., as an asymptote, $X(t\to\infty)\Rightarrow X\to\eta$ because $De\to 0$. The Deborah number through the factor $1/\sqrt{(1+De)}$ reduces η for short times (see Figure 3-(d), thus in the fluid-like range where the viscous shear pulse propagation are enhanced. The factor $1/\sqrt{(1+De)}$ can also be interpreted as a relaxation function: it goes to 1 for fluid-like behavior ($De\to 0$) and 0 for the solid-like behavior ($De\to\infty$).

4.2.1. Deborah number: more comments

Taking into account that the time ratio $\beta/\nu t$ appears naturally in the expressions of the penetration depths, we refer to the elasticity number, El defined as $El=\alpha/(L/\nu)$, where L/ν is the Newtonian diffusive macroscopic time scale. Now, assuming that the length scale is defined as $L=\sqrt{\nu t}$ in the case of viscoelastic flow with the time-fractional model, as the more general one, we get $El=\alpha/t$. In this context,

$$De=\beta/\nu t=\left(\sqrt{\beta}/\sqrt{\nu t}\right)=L/L \tag{23}$$

where $L=\sqrt{\beta}$. Additionally, we can express β as $\beta=\nu\lambda$ and get $L=\sqrt{\nu\lambda}$.

Furthermore, we can express the Deborah number as a ratio of two length scales $De=L_{\beta}/L_{\nu}$. Precisely, as a proportion of the depths of penetration of the impulse is imposed at the boundary and transported into the fluid depth by two mechanisms: *the viscous shear* (L) and *the elastic response* (L). Small De numbers because reason of low relaxation times, λ or as time goes on, shift the fluid flow toward the Newtonian one. Otherwise, long propagation in depth of the elastic pulse L and short viscous shear propagation length L correspond to high Deborah numbers, and *vice versa*. Otherwise, with an increase β caused by large relaxation times λ or due to short L (high viscosity μ), the fluid behaves as a solid. In any case, all these modes depend on the timescale chosen, which is the crux in the definition of the Deborah number.

In addition to the comments on the physical meaning of the Deborah number $De = \beta / \nu t = \left(\sqrt{\beta} / \sqrt{\nu t} \right)$, it can be considered as a similarity variable with a length scale $\sqrt{\nu t}$ and a fixed space coordinate $\sqrt{\beta}$. Hence, the approximate velocity profile is a function of two similarity variables: $\eta = y / \sqrt{\nu t}$ and $\chi = \sqrt{\beta} / \sqrt{\nu t} = \sqrt{De}$. The former corresponds to the Newtonian flow, while the second similarity variable corresponds to the elastic mode of momentum transport ($\chi = De$), which depends on the fractional order γ and decays in time.

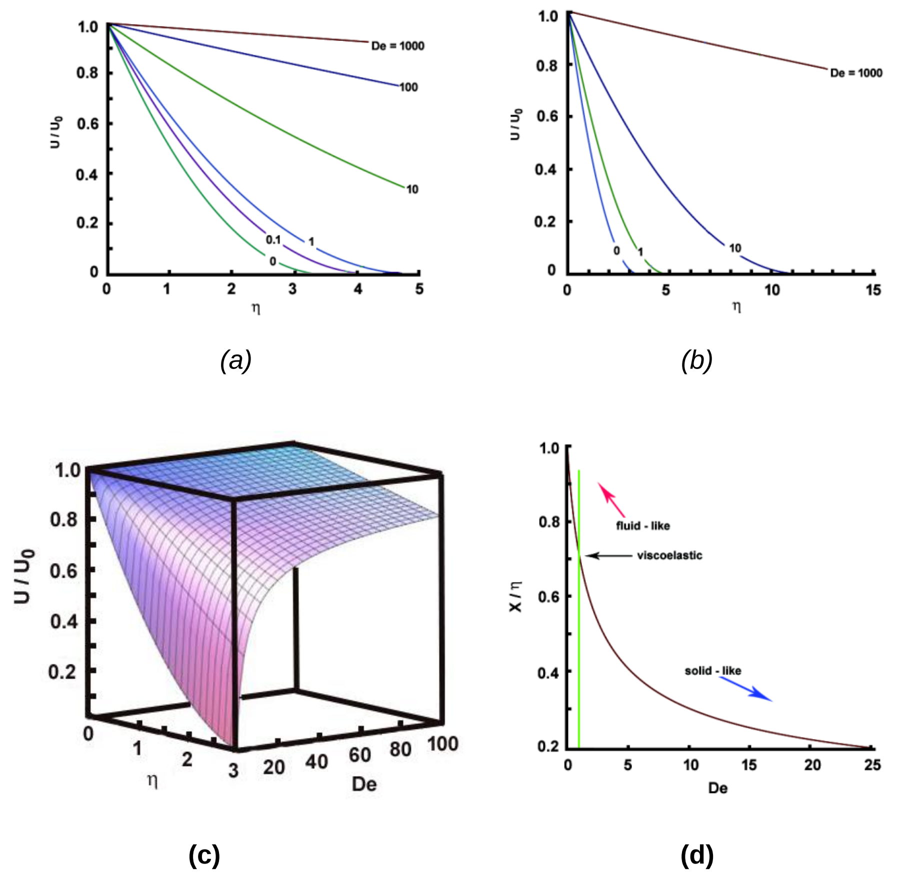

Figure 4. *Velocity profiles vs. the similarity variable η and the Deborah number as a parameter: (a) and (b); (c)-three-dimensional plot; (d) the variation of the factor X as function of the the Deborah number:* ***Note:*** *For the sake of simplicitty, we chose $n = 2$ in the simulations.*

4.3. A solution controlled by the *Strouhal number*

Applying the double integration to the integer-order model (S2-2b)

$$\iint \frac{\partial u}{\partial \tilde{t}} dy dy = \iint \frac{\partial u}{\partial \tilde{y}} dy dy + St \iint \frac{\partial}{\partial \tilde{t}} \left(\frac{\partial u}{\partial \tilde{y}} \right) dy dy \tag{24}$$

And, by doing what the integral method needs, we obtain for the integer-order model,

$$\tilde{\delta}(\tilde{t}) = \sqrt{\tilde{t}} \sqrt{\frac{Wi}{\mathrm{Re}}} \sqrt{N} = \sqrt{\tilde{t}} \sqrt{1 + St} \sqrt{N}, \quad N = (n+1)(n+2). \tag{25}$$

Here, we express the dimensionless penetration depth $\tilde{\delta}(\tilde{t}) = \delta / \sqrt{\nu t}$ because we consider a semi-infinite fluid, and $L = \sqrt{\nu t}$ is the length scale in such cases.

Hence, the approximate velocity profile is

$$\frac{u}{U} = \left(1 - \frac{\tilde{y}}{\sqrt{N}\sqrt{1+St}}\right) = \left(1 - \frac{y}{\sqrt{\nu t}}\frac{1}{\sqrt{N}\sqrt{1+St}}\right) = \left(1 - \frac{\eta}{\sqrt{N}\sqrt{1+St}}\right), \ \eta = \frac{y}{\sqrt{\nu t}} . \tag{26}$$

This resembles the results obtained using the Deborah number.

5. The outcomes of the approximate solutions

The solutions developed clearly defined two dimensionless variables: the Boltzmann similarity variable η and the Deborah number De (alternatively the Strouhal number St). Hence, we may generally formulate functional relationships,

$$u = \frac{U}{U} = f(\eta, De), \ u = \frac{U}{U} = f(\eta, St). \tag{27}$$

Alternatively using $X(De) = \dfrac{\eta}{\sqrt{(1+De)}}$ or $X(St) = \dfrac{\eta}{\sqrt{(1+St)}}$

Thus, relying on the relaxation function $(1+De)$ as

$$u = \frac{U}{U} = f(X(De)), \ u = \frac{U}{U} = f(X(St)) = f\left(X\left(\frac{Wi}{\mathrm{Re}}\right)\right). \tag{28}$$

The reason for using this $X(St)$ is that as time goes on, the variable X approaches the Boltzmann similarity variable η, that is $X(t \to \infty) \equiv X(De \to 0) \Rightarrow \eta$, and the flow approaches the Newtonian one.

These general formulations are consistent with the solution provided by Irgens [18] (Chapter 7 in [18]), where for the shear stress in a Maxwell fluid, we have

$$\tau(t) = \tau\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right] \Rightarrow \tau(t) = \tau\left[1 - \exp\left(-\frac{1}{De}\right)\right] \tag{29}$$

where $\tau = \tau(\infty) = -\mu \dfrac{dU}{dy}$ is the Newtonian shear stress.

Hence,

$$\tau(t) = \tau\left[1 - \exp\left(-\frac{1}{De}\right)\right] \Rightarrow \begin{cases} \tau \to \tau, & De \to 0 \\ \tau \to 0, & De \to \infty \end{cases} \tag{30}$$

This can be easily illustrated by the plot in Figure 5, showing $\tau(t)/\tau = f(De)$, which to a greater extent, consistency with the behavior of the variable $X(De)$ presented in Figure 4(d).

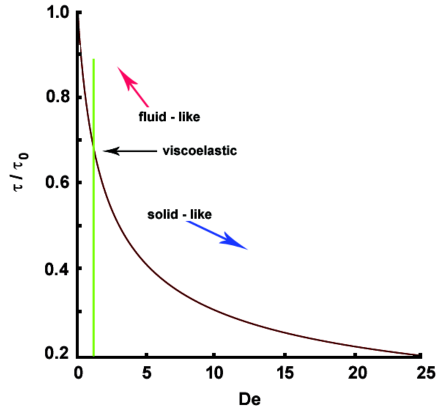

Figure 5. The relaxation function from Eq. (28).

The stress field $\tau(y,t)$ can be easily calculated from the general model (28) and the approximate solution (29), respectively.

$$\tau = \nu \frac{\partial u}{\partial y} + \beta \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) \Rightarrow \tau(y,t) = \nu \frac{n}{\delta}\left(1-\frac{y}{\delta}\right) + \beta \frac{n(n-1)}{\delta}\left(1-\frac{y}{\delta}\right) \frac{y}{\delta}\frac{d\delta}{dt} \tag{31}$$

And at the driven interface $y = 0$ with velocity $U = U$

$$\tau(t, y=0) = \nu \frac{n}{\delta} = \sqrt{\frac{\nu}{t}} \frac{n}{\sqrt{(n+1)(n+2)}\sqrt{1+\frac{\beta}{\nu t}}} = \sqrt{\frac{\nu}{t}} \frac{n}{\sqrt{(n+1)(n+2)}\sqrt{1+De}} \tag{32}$$

and we can see the effect of the term $\frac{1}{\sqrt{1+De}}$.

Particularly, for $De = 0$ we have

$$\tau(t, y=0)\big| = \sqrt{\frac{\nu}{t}} \frac{n}{\sqrt{(n+1)(n+2)}} \tag{33}$$

6. A discrete Stokes' first problem: A finite fluid volume spreading over a horizontal surface

We will now address the issue initially presented (see Figure 1). Following the scaling and approximate solutions, we identified the dimensionless groups that govern fluid flow. We aim to determine the outcomes of the discriminant dimensional analysis when a discrete self-preserving fluid volume is subjected to the same boundary conditions as in Stokes' first problem and how the similarities in the boundary layer will manifest.

Consider a finite rectangular volume of viscoelastic fluid, initially at rest, with a height H and length L, preserving its volume during the deformation, i.e., $A = H L = HL = const.$, as presented in Figure 1.

The dimensional analysis reveals the following set of physical quantities.

$$\frac{H}{L} \sim t^a U_o^b \beta^c \tag{34}$$

Then

$$L_x^{-1} L_y = T^{a+d} \left(L_x T^{-1} \right)^b \left(L_y^2 T^{-1} \right)^c \tag{35}$$

where,

$L \to -1 = b$, $L \to 1 = 2c$, $T \to 0 = a - b$ then $a = -1$, $b = -1$, $c = 1/2$.

Thus, we get,

$$\frac{H}{L} \sim t^{-1} U_o^{-1} \beta^{1/2} \Rightarrow \frac{H}{L} \sim \left(\frac{U_o t}{\beta^{1/2}} \right)^{-1} \tag{36}$$

$$\frac{H}{L} \sim \left(\frac{U_o t}{\beta^{1/2}} \times \frac{\beta^{1/2}}{\beta^{1/2}} \right)^{-1} \sim \left(\frac{t}{\lambda} \times \frac{U_o \beta^{1/2}}{\nu} \right)^{-1} \tag{37}$$

$$\frac{H}{L} \sim \left(\frac{t}{\lambda} \right)^{-1} \mathrm{Re}_\beta^{-1} \sim \frac{1}{De} \frac{1}{\mathrm{Re}_\beta} \ . \tag{38}$$

If $t / \lambda \gg 1$, i.e., for $De \gg 1$ we have the Newtonian approximation,

$$\frac{H}{L} \sim \mathrm{Re}_\beta^{-1} \tag{39}$$

and $H/L \to 0$, i.e., as in an infinitely thin layer.

Conclusion

The scaling and approximate solution of the second-grade viscoelastic fluid in Stokes' first problem, combined with dimensional analysis, allowed the dimensionless group to effectively control the development of the flow field. As a main outcome, the identification of the Deborah number was found, and its detailed analysis yielded a physical analysis of the transition from elastic through viscoelastic towards Newtonian viscous flow. The dimensional analysis of a discrete viscoelastic fluid volume sheared by a constant velocity at the top interface reveals the main controlling dimensionless groups and allows us to see the asymptote as a thin fluid layer with thickness controlled by the inverse of the Reynolds number, a result consistent with the result from the classical boundary layer flow.

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